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THE FOUNDATIONS OF MATHEMATICS.

Principia Mathematica. By Dr. A. N. Whitehead, F.R.S., and B. Russell, F.R.S. Vol. i. Pp. xv+666. (Cambridge: University Press, 1910.) Price 25*s.* net.

THIS work contains some thousands of propositions, each, with its proof, expressed in a shorthand so concise that if they were all expanded into ordinary language, the room taken up would be ten times as large at least; space, time, and mass are not considered at all, and arithmetic is merely foreshadowed by the introduction of the symbols 0, 1, 2, and 2_r. How then, it may be asked, can the authors pretend to be writing about mathematics? The answer amounts to saying that for every branch of the tree of knowledge there is a corresponding root, and every advance in climbing seems to compel a similar advance in delving. Just as the discovery of non-Euclidean geometries led to the reconsideration of geometrical axioms, so Cantor's invention of transfinite numbers has reacted upon the theory of elementary arithmetic, and hence upon the whole of analysis and all its applications.

Besides this, there has grown up a school of mathematicians intensely interested in the logical side of their subject. Indeed, this was inevitable as soon as the primary distinction between ordinal and cardinal number was fully grasped, and the nature of the arithmetical continuum had been strictly defined. The inquirer was driven back and back to questions of order, and correspondence, and relations, and classes, until he felt bound to construct a symbolical logic fit to express the chain of deductions he found latent in the most familiar processes of arithmetic. This has led to an immense aggregation of what may be called mathematical prolegomena; and with this the first volume of the "Principia Mathematica" is almost exclusively concerned.

Thus the actual titles of its two parts are "Mathematical Logic" and "Prolegomena to Cardinal Arithmetic," and both are so elaborate that only a meagre account of them can be given in a review. The theory of deduction is based upon seven assumptions, called primitive propositions, and upon the notions of disjunction (p or q) and implication (either not-p or q). In about thirty pages the authors obtain the main results of the purely formal logic of propositions. This is followed by a very interesting section on "apparent variables," including the theory of propositions of different orders. A real advance seems to have been made here in the analysis of vicious-circle fallacies, and false generalisations, especially as they occur in mathematical reasoning. It is pointed out that such phrases as "all propositions" or "all properties of x" are strictly meaningless, and a legitimate use of such terms is based upon an axiom of reducibility (pp. 173–5) which is stated in the form: "Any function of one argument or of two is formally equivalent to a predicative function of the same argument or arguments," and its main use is at the beginning of the calculus of classes (p. 197). Whether this axiom is really simpler than the introduction of "class" as a primitive term seems debatable, but it does not matter much for practical purposes.

The next three sections deal with classes and relations, and introduce a large number of new symbols and a long series of propositions. Fortunately here, as elsewhere, each section is preceded by a summary, giving the principal theorems; and, in fact, the reader will find it helpful to go through all these summaries (after the introduction) before attacking the chapters in detail.

Coming now to the more directly mathematical part, we have first of all (p. 356) a discussion of unit classes, which illustrates the subtleties of this new calculus. Thus it is found necessary to construct a symbol for "the class of which the only member is x," as distinguished from x itself. At first this seems to be superfluous, but when we suppose x to be a class, we see that it is not. The next step is to define the cardinal number 1 as the class of all unit classes. Similarly the cardinal number 2 is defined as the class of all couples (x, y) such that (x, y) and (y, x) are equivalent; and the ordinal number 2_r as the class of ordered couples (x, y) such that (y, x) is different from (x, y). Besides these we have a symbol $\dot{2}$ for the class of all relations consisting of a single couple, including couples (x, x). Then we have a series of theorems on subclasses, relative types of classes, one-one and one-many relations, &c., leading up to the fundamental notion of similarity of classes which is the necessary basis of all arithmetic proper.

We next come to the difficult question of selections, from relations and from classes of classes respectively.

"If k is a class of classes, then μ is called a selected class of k when μ is formed by choosing one term out of each member of k."

(It would perhaps be more precise to say "a class selected from k," because μ, as a class, is not generally a member of k.) Now at first sight it looks as if a selected class could always be formed, but this is not really obvious when k is infinite, and, in fact, it has not been proved in general. If it could be, it would follow that every class can be well-ordered, and the difficulty of asserting this in general can be seen from a special case. Consider the aggregate of colours, merely as sensations of my own; how can I order them, without importing some additional foreign element, such as the time when I first became conscious of a particular one, or its analysis by a colour-box, or something of that sort? Besides this, there is the logical difficulty of making an assertion about "every" class, for one reason because assertions form a class.[1] Hence the section (p. 561) on the conditions for the existence of selections is one of special interest: its most important bearing on arithmetic is in the theory of multiplication.

The final section is on inductive relations, especially

[1] This is undeniable, because "assertions do not form a class" is itself an assertion, and only a formal, not a real contradiction of the above statement.

the ancestral relation, which, to avoid a vicious circle, is defined so as to apply to members of an infinite class. As the authors explain in a note, this section is mainly based upon Frege's work, and is used afterwards to deduce the properties of finite cardinals and the transfinite cardinal $\aleph_0$. Here we find the Peanesque notation in all its development, and must make up our minds to learn it thoroughly, or else to express its formulæ in an equally exact, but less unfamiliar symbolism. This leads us to the few critical remarks that we venture to offer on this admirable and elaborate work. Every communication of ideas from one mind to another is made by means of a conventional symbolism; no symbolism can be more *exact* than language, because language is, in the last resort, required to explain and define it. But it may be more *concise* than language, and this is the real virtue of the Peano notation and its derivatives. To show how easy it is to exaggerate the value of the notation as such, we may take an example from p. 16 of the present work. The authors say that "it is an obvious error, though one easy to commit," that "No A is B" is the contradictory of "every A is B," and proceed to add that the symbolism exposes the fallacy at once. Really it does nothing of the kind; truly the symbol $\sim \{(x).\phi(x)\}$, the contradictory of $(x).\phi(x)$, is different *in form* from $(x). \sim \phi(x)$, but how can we tell from looking at them that these last two symbols are not equivalent? Again, the authors profess to give a proof of the law of excluded middle; they assume it in defining the assertion symbol, for they practically say "if the proposition to which this sign is prefixed is false the book is in error," tacitly assuming (here) that all their propositions are significant. The law of excluded middle is surely axiomatic for a significant proposition, the only trouble is in being quite sure that our assertions are really significant, and in this it is reason that must guide us, not symbolism, though a proper choice of symbolism may conduce to economy of thought. As an illustration, we may refine a little on the paradox of Epimenides. Suppose a Frenchman or a German asserts "Every statement that has ever been made by an Englishman is false." This is a significant statement, and *as such* must be true or false. But suppose an Englishman says the same thing: the proposition ceases to be significant, unless he adds "except one," when it again becomes significant. Questions of this kind are not so trivial as they appear, and a really philosophical study of language might do a good deal towards making more definite the metaphysical basis of knowledge.

G. B. M.

MOVEMENT AND ESCAPEMENT.

Le Mouvement. Mesures de l'étendue et mesures du temps. By Prof. J. Andrade. Pp. vi+328. (Paris: Librairie Félix Alcan, 1911.) Price 6 francs.

SOME literary effusions—for instance, the novel with a purpose—present to the reviewer an awkward problem, namely, whether to concentrate his attention on the novel as such, or on the purpose. The present work might almost be included in some such category, inasmuch as it may be regarded from the point of view of a mathematician pure and simple, of a more or less practical mechanic, or even of an astronomer, while all the time it apparently claims to be a philosophical treatise, and as such to appeal to what may be called the general reader. In some parts of the book the philosopher is much in evidence, and in many places the absence of diagrams, and the assumption that the reader will understand determinants, vectors, or even ordinary equations of motion without explanation, would certainly repel the ordinary reader. The mathematician will find perhaps little that is novel. The suggestions of non-Euclidean space, whether that of Lobatchewsky or of Riemann, are little more than suggestions, and can only give those to whom such ideas are new the kind of shock the earlier cyclists felt on first riding a free-wheel. On the other hand, a very good historical sketch, amply provided with diagrams, is given of the development of scientific clock-making with due respect to the great English horologists.

A brief sketch of the contents of the book will serve to indicate the scope of the author's endeavour, and it is difficult to conceive how, within the limits of such a volume, a perfectly satisfactory result could have been achieved. Perhaps only a fellow-countryman of the great French philosophers of the past would ever have attempted such a task. The first part treats of geometrical ideas of number and space, the author showing a decided preference for vectorial or polar coordinates, and for rotation as a means of translation. The finite straight line is elaborately discussed, and ordinary geometrical propositions regarded from the point of view that came into vogue about a quarter of a century ago, when Nixon's Euclid began to oust Todhunter's in some schools. Triangles and solids, plane and spherical areas, volumes, velocity, vectors, the theorems of Ampère and Stokes, moments, composition of vectors and vectorial quantities, bring us through trigonometry and statics to non-Euclidean geometry by a somewhat tortuous route.

The second part introduces force, one chapter being devoted to the notions of astronomy and celestial mechanics from Hipparchus to Newton, and another to the principles of dynamics, equilibrium, and the two fundamentals, which, in the author's view, are clock and orientation; a third dwells on the vital importance of a function of forces, on stability and conservative systems, on isolated systems, Painlevé's theorem and Laplace's invariable plane; and these are followed by simple and damped oscillations, spiral movement, elastic bodies, and fluids, and the bending of springs. The third part deals with optics more especially of the telescope, with a more special devotion to different methods of geodesy from Picard to the use of invar and the Jäderin wire, and the correction of the units of the metric system.

The fourth and last part deals with the chronometer in general, and escapements in particular, calling for continued experimental work on indicated